

SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR TWO OPERATOR CONVEX FUNCTIONS

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ABSTRACT. In this paper we establish some new Hermite-Hadamard type inequalities for two operator convex functions of selfadjoint operators in Hilbert spaces.

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1. INTRODUCTION

The following inequality holds for any convex function f defined on \mathbb{R} and $a, b \in \mathbb{R}$, with $a < b$

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \rightarrow \mathbb{R}$.

Let X be a vector space, $x, y \in X$, $x \neq y$. Define the segment

$$[x, y] := \{(1-t)x + ty : t \in [0, 1]\}.$$

We consider the function $f : [x, y] \rightarrow \mathbb{R}$ and the associated function

$$\begin{aligned} g(x, y) &: [0, 1] \rightarrow \mathbb{R}, \\ g(x, y)(t) &:= f((1-t)x + ty), t \in [0, 1]. \end{aligned}$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$. For any convex function defined on a segment $[x, y] \subseteq X$, we have the Hermite-Hadamard integral inequality

$$(1.2) \quad f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x + ty)dt \leq \frac{f(x)+f(y)}{2},$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x, y) : [0; 1] \rightarrow \mathbb{R}$.

In recent years several extensions and generalizations have been considered for classical convexity. A number of papers have been written on this inequality providing some inequalities analogous to Hadamard's inequality given in (1.1) involving two convex functions, see [5, 6]. The main purpose of this paper is to establish some new integral inequalities for two operator convex functions of selfadjoint operators in Hilbert spaces which generalize Theorem 1 in [5] and Theorem 3 in [6].

In order to do that we need the following preliminary definitions and results. Let A be a bounded self adjoint linear operator on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. The Gelfand map establishes a $*$ -isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of A , denoted $Sp(A)$, and the C^* -algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [3, p.3]). For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

- (i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;
- (ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f^*) = \Phi(f)^*$;
- (iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;
- (iv) $\Phi(f_0) = 1$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the continuous functional calculus for a bounded selfadjoint operator A . If A is a bounded selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e., $f(A)$ is a positive operator on H . Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$(P) \quad f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A)$$

in the operator order in $B(H)$.

A real valued continuous function f on an interval I is said to be operator convex (operator concave) if

$$(OC) \quad f((1 - \lambda)A + \lambda B) \leq (\geq) (1 - \lambda)f(A) + \lambda f(B)$$

in the operator order in $B(H)$, for all $\lambda \in [0, 1]$ and for every bounded selfadjoint operators A and B in $B(H)$ whose spectra are contained in I .

Dragomir in [2] has proved a Hermite-Hadamard type inequality for operator convex function as follows:

Theorem 1. *Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval I . Then for any selfadjoint operators A and B with spectra in I , the following inequality holds*

$$\begin{aligned} \left(f\left(\frac{A+B}{2}\right) \leq \right) & \frac{1}{2} \left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right) \right] \\ & \leq \int_0^1 f((1-t)A + tB) dt \\ & \leq \frac{1}{2} \left[f\left(\frac{A+B}{2}\right) + \frac{f(A) + f(B)}{2} \right] \left(\leq \frac{f(A) + f(B)}{2} \right). \end{aligned}$$

A generalization of Theorem 1 for operator preinvex functions of selfadjoint operators in Hilbert spaces is established in [4], as follows:

Theorem 2. *Let $S \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : S \times S \rightarrow B(H)_{sa}$ and η satisfies condition C. If for every $A, B \in S$ and $V = A + \eta(B, A)$ the function $f : I \rightarrow \mathbb{R}$ is operator preinvex with respect to η on η -path P_{AV} with spectra of A and spectra of V in the interval I . Then the following inequality holds*

$$\begin{aligned} f\left(\frac{A+V}{2}\right) & \leq \frac{1}{2} \left[f\left(\frac{3A+V}{4}\right) + f\left(\frac{A+3V}{4}\right) \right] \\ (1.3) \quad & \leq \int_0^1 f(A + t\eta(B, A)) dt \\ & \leq \frac{1}{2} \left[f\left(\frac{A+V}{2}\right) + \frac{f(A) + f(V)}{2} \right] \leq \frac{f(A) + f(B)}{2}. \end{aligned}$$

2. THE MAIN RESULTS

Let $f, g : I \rightarrow \mathbb{R}$ be operator convex functions on the interval I . Then for any selfadjoint operators A and B on a Hilbert space H with spectra in I , we define real functions $M(A, B)$, $N(A, B)$ and $P(A, B)$ on H by

$$\begin{aligned} M(A, B)(x) &= \langle f(A)x, x \rangle \langle g(A)x, x \rangle + \langle f(B)x, x \rangle \langle g(B)x, x \rangle \quad (x \in H), \\ N(A, B)(x) &= \langle f(A)x, x \rangle \langle g(B)x, x \rangle + \langle f(B)x, x \rangle \langle g(A)x, x \rangle \quad (x \in H), \\ P(A, B)(x) &= \langle [f(A)g(A) + f(B)g(B)]x, x \rangle \quad (x \in H). \end{aligned}$$

Lemma 1. *Let $f : I \rightarrow \mathbb{R}$ be a continuous function on the interval I . Then for every two selfadjoint operators A, B with spectra in I*

the function f is operator convex on $[A, B]$ if and only if the function $\varphi_{x,A,B} : [0, 1] \rightarrow \mathbb{R}$ defined by

$$(2.1) \quad \varphi_{x,A,B} = \langle f((1-t)A + tB)x, x \rangle$$

is convex on $[0, 1]$ for every $x \in H$ with $\|x\| = 1$.

Proof. Let f be operator convex on $[A, B]$ then for any $t_1, t_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ we have

$$\begin{aligned} \varphi_{x,A,B}(\alpha t_1 + \beta t_2) &= \langle f((1 - (\alpha t_1 + \beta t_2))A + (\alpha t_1 + \beta t_2)B)x, x \rangle \\ &= \langle f(\alpha[(1 - t_1)A + t_1B] + \beta[(1 - t_2)A + t_2B])x, x \rangle \\ &\leq \alpha \langle f((1 - t_1)A + t_1B)x, x \rangle + \beta \langle f((1 - t_2)A + t_2B)x, x \rangle \\ &= \alpha \varphi_{x,A,B}(t_1) + \beta \varphi_{x,A,B}(t_2). \end{aligned}$$

showing that $\varphi_{x,A,B}$ is a convex function on $[0, 1]$.

Let now $\varphi_{x,A,B}$ be convex on $[0, 1]$, we show that f is operator convex on $[A, B]$. For every $C = (1 - t_1)A + t_1B$ and $D = (1 - t_2)A + t_2B$ we have

$$\begin{aligned} \langle f(\lambda C + (1 - \lambda)D)x, x \rangle &= \langle f[\lambda((1 - t_1)A + t_1B) + (1 - \lambda)((1 - t_2)A + t_2B)]x, x \rangle \\ &= \langle f[(1 - (\lambda t_1 + (1 - \lambda)t_2))A + (\lambda t_1 + (1 - \lambda)t_2)B]x, x \rangle \\ &= \varphi_{x,A,B}(\lambda t_1 + (1 - \lambda)t_2) \leq \lambda \varphi_{x,A,B}(t_1) + (1 - \lambda) \varphi_{x,A,B}(t_2) \\ &= \lambda \langle f((1 - t_1)A + t_1B)x, x \rangle + (1 - \lambda) \langle f((1 - t_2)A + t_2B)x, x \rangle \\ &\leq \lambda \langle f(C)x, x \rangle + (1 - \lambda) \langle f(D)x, x \rangle. \end{aligned}$$

□

Theorem 3. Let $f, g : I \rightarrow \mathbb{R}^+$ be operator convex functions on the interval I . Then for any selfadjoint operators A and B on a Hilbert space H with spectra in I , the inequality

$$(2.2) \quad \int_0^1 \langle f(tA + (1 - t)B)x, x \rangle \langle g(tA + (1 - t)B)x, x \rangle dt \leq \frac{1}{3}M(A, B)(x) + \frac{1}{6}N(A, B)(x).$$

holds for any $x \in H$ with $\|x\| = 1$.

Proof. For $x \in H$ with $\|x\| = 1$ and $t \in [0, 1]$, we have

$$(2.3) \quad \langle (tA + (1 - t)B)x, x \rangle = t \langle Ax, x \rangle + (1 - t) \langle Bx, x \rangle \in I,$$

since $\langle Ax, x \rangle \in Sp(A) \subseteq I$ and $\langle Bx, x \rangle \in Sp(B) \subseteq I$.

Continuity of f, g and (2.3) imply that the operator valued integrals $\int_0^1 f(tA + (1-t)B)dt$, $\int_0^1 g(tA + (1-t)B)dt$ and $\int_0^1 (fg)(tA + (1-t)B)dt$ exist.

Since f and g are operator convex, therefore for t in $[0, 1]$ and $x \in H$ we have

$$(2.4) \quad \langle f(tA + (1-t)B)x, x \rangle \leq \langle (tf(A) + (1-t)f(B))x, x \rangle,$$

$$(2.5) \quad \langle g(tA + (1-t)B)x, x \rangle \leq \langle (tg(A) + (1-t)g(B))x, x \rangle.$$

From (2.4) and (2.5) we obtain

$$(2.6) \quad \begin{aligned} & \langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle \\ & \leq t^2 \langle f(A)x, x \rangle \langle g(A)x, x \rangle + (1-t)^2 \langle f(B)x, x \rangle \langle g(B)x, x \rangle \\ & \quad + t(1-t) [\langle f(A)x, x \rangle \langle g(B)x, x \rangle + \langle f(B)x, x \rangle \langle g(A)x, x \rangle]. \end{aligned}$$

Integrating both sides of (2.6) over $[0, 1]$ we get the required inequality (2.2). \square

Theorem 4. *Let $f, g : I \rightarrow \mathbb{R}$ be operator convex functions on the interval I . Then for any selfadjoint operators A and B on a Hilbert space H with spectra in I , the inequality*

$$(2.7) \quad \begin{aligned} & \left\langle f\left(\frac{A+B}{2}\right)x, x \right\rangle \left\langle g\left(\frac{A+B}{2}\right)x, x \right\rangle \\ & \leq \frac{1}{2} \int_0^1 \langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle dt \\ & \quad + \frac{1}{12}M(A, B)(x) + \frac{1}{6}N(A, B)(x), \end{aligned}$$

holds for any $x \in H$ with $\|x\| = 1$.

Proof. Since f and g are operator convex, therefore for any $t \in I$ and any $x \in H$ with $\|x\| = 1$ we observe that

$$\begin{aligned} & \left\langle f\left(\frac{A+B}{2}\right)x, x \right\rangle \left\langle g\left(\frac{A+B}{2}\right)x, x \right\rangle \\ & = \left\langle f\left(\frac{tA + (1-t)B}{2} + \frac{(1-t)A + tB}{2}\right)x, x \right\rangle \\ & \quad \times \left\langle g\left(\frac{tA + (1-t)B}{2} + \frac{(1-t)A + tB}{2}\right)x, x \right\rangle \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{4}[\langle f(tA + (1-t)B)x, x \rangle + \langle f((1-t)A + tB)x, x \rangle] \\
&\quad \times [\langle g(tA + (1-t)B)x, x \rangle + \langle g((1-t)A + tB)x, x \rangle] \\
&\leq \frac{1}{4}[\langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle \\
&\quad + \langle f((1-t)A + tB)x, x \rangle \langle g((1-t)A + tB)x, x \rangle] \\
&+ \frac{1}{4}[t\langle f(A)x, x \rangle + (1-t)\langle f(B)x, x \rangle][(1-t)\langle g(A)x, x \rangle + t\langle g(B)x, x \rangle] \\
&+ [(1-t)\langle f(A)x, x \rangle + t\langle f(B)x, x \rangle][t\langle g(A)x, x \rangle + (1-t)\langle g(B)x, x \rangle] \\
&= \frac{1}{4}[\langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle \\
&\quad + \langle f((1-t)A + tB)x, x \rangle \langle g((1-t)A + tB)x, x \rangle] \\
&\quad + \frac{1}{4}2t(1-t)[\langle f(A)x, x \rangle \langle g(A)x, x \rangle + \langle f(B)x, x \rangle \langle g(B)x, x \rangle] \\
&\quad + (t^2 + (1-t)^2)[\langle f(A)x, x \rangle \langle g(B)x, x \rangle + \langle f(B)x, x \rangle \langle g(A)x, x \rangle].
\end{aligned}$$

Therefore we get

$$\begin{aligned}
(2.8) \quad &\left\langle f\left(\frac{A+B}{2}\right)x, x \right\rangle \left\langle g\left(\frac{A+B}{2}\right)x, x \right\rangle \\
&\leq \frac{1}{4}[\langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle \\
&\quad + \langle f((1-t)A + tB)x, x \rangle \langle g((1-t)A + tB)x, x \rangle] \\
&\quad + \frac{1}{4}2t(1-t)[\langle f(A)x, x \rangle \langle g(A)x, x \rangle + \langle f(B)x, x \rangle \langle g(B)x, x \rangle] \\
&\quad + (t^2 + (1-t)^2)[\langle f(A)x, x \rangle \langle g(B)x, x \rangle + \langle f(B)x, x \rangle \langle g(A)x, x \rangle].
\end{aligned}$$

We integrate both sides of (2.8) over $[0,1]$ and obtain

$$\begin{aligned}
&\left\langle f\left(\frac{A+B}{2}\right)x, x \right\rangle \left\langle g\left(\frac{A+B}{2}\right)x, x \right\rangle \\
&\leq \frac{1}{4} \int_0^1 [\langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle \\
&\quad + \langle f((1-t)A + tB)x, x \rangle \langle g((1-t)A + tB)x, x \rangle] dt \\
&\quad + \frac{1}{12}M(A, B)(x) + \frac{1}{6}N(A, B)(x).
\end{aligned}$$

This implies the required inequality (2.7). \square

Theorem 5. Let $f, g : I \rightarrow \mathbb{R}$ be operator convex functions on the interval I . Then for any selfadjoint operators A and B on a Hilbert space H with spectra in I , we have the inequality

$$\begin{aligned}
 (2.9) \quad & \left\langle f\left(\frac{A+B}{2}\right)x, x\right\rangle \int_0^1 \langle g(tA + (1-t)B)x, x\rangle dt \\
 & + \left\langle g\left(\frac{A+B}{2}\right)x, x\right\rangle \int_0^1 \langle f(tA + (1-t)B)x, x\rangle dt \\
 & \leq \frac{1}{2} \int_0^1 \langle f(tA + (1-t)B)x, x\rangle \langle g(tA + (1-t)B)x, x\rangle dt \\
 & + \frac{1}{12}M(A, B)(x) + \frac{1}{6}N(A, B)(x) + \left\langle f\left(\frac{A+B}{2}\right)x, x\right\rangle \left\langle g\left(\frac{A+B}{2}\right)x, x\right\rangle.
 \end{aligned}$$

Proof. Since f and g are operator convex, then for $t \in [0, 1]$ we observe that

$$\begin{aligned}
 \left\langle f\left(\frac{A+B}{2}\right)x, x\right\rangle &= \left\langle f\left(\frac{tA + (1-t)B}{2} + \frac{(1-t)A + tB}{2}\right)x, x\right\rangle \\
 &\leq \left\langle \frac{f(tA + (1-t)B) + f((1-t)A + tB)}{2}x, x\right\rangle \\
 \left\langle g\left(\frac{A+B}{2}\right)x, x\right\rangle &= \left\langle g\left(\frac{tA + (1-t)B}{2} + \frac{(1-t)A + tB}{2}\right)x, x\right\rangle \\
 &\leq \left\langle \frac{g(tA + (1-t)B) + g((1-t)A + tB)}{2}x, x\right\rangle.
 \end{aligned}$$

we multiply by one under the other and by one across the other of the above inequality and then we add these inequalities, so we obtain

$$\begin{aligned}
 & \left\langle f\left(\frac{A+B}{2}\right)x, x\right\rangle \left\langle \frac{g(tA + (1-t)B) + g((1-t)A + tB)}{2}x, x\right\rangle \\
 & + \left\langle g\left(\frac{A+B}{2}\right)x, x\right\rangle \left\langle \frac{f(tA + (1-t)B) + f((1-t)A + tB)}{2}x, x\right\rangle \\
 & \leq \left\langle \frac{f(tA + (1-t)B) + f((1-t)A + tB)}{2}x, x\right\rangle \\
 & \quad \times \left\langle \frac{g(tA + (1-t)B) + g((1-t)A + tB)}{2}x, x\right\rangle \\
 & \quad + \left\langle f\left(\frac{A+B}{2}\right)x, x\right\rangle \left\langle g\left(\frac{A+B}{2}\right)x, x\right\rangle.
 \end{aligned}$$

This implies that

$$\begin{aligned}
& \frac{1}{2} \left\langle f \left(\frac{A+B}{2} \right) x, x \right\rangle [\langle g(tA + (1-t)B)x, x \rangle + \langle g((1-t)A + tB)x, x \rangle] \\
& + \frac{1}{2} \left\langle g \left(\frac{A+B}{2} \right) x, x \right\rangle [\langle f(tA + (1-t)B)x, x \rangle + \langle f((1-t)A + tB)x, x \rangle] \\
& \leq \frac{1}{4} [\langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle \\
& + \langle f((1-t)A + tB)x, x \rangle \langle g((1-t)A + tB)x, x \rangle] \\
& + \frac{1}{4} [t \langle f(A)x, x \rangle + (1-t) \langle f(B)x, x \rangle] [(1-t) \langle g(A)x, x \rangle + t \langle g(B)x, x \rangle] \\
& + \frac{1}{4} [(1-t) \langle f(A)x, x \rangle + t \langle f(B)x, x \rangle] [t \langle g(A)x, x \rangle + (1-t) \langle g(B)x, x \rangle] \\
& + \left\langle f \left(\frac{A+B}{2} \right) x, x \right\rangle \left\langle g \left(\frac{A+B}{2} \right) x, x \right\rangle.
\end{aligned}$$

Again integration both side of the above inequality over $[0, 1]$ and obtain

$$\begin{aligned}
& \frac{1}{2} \left\langle f \left(\frac{A+B}{2} \right) x, x \right\rangle \int_0^1 [\langle g(tA + (1-t)B)x, x \rangle + \langle g((1-t)A + tB)x, x \rangle] dt \\
& + \frac{1}{2} \left\langle g \left(\frac{A+B}{2} \right) x, x \right\rangle \int_0^1 [\langle f(tA + (1-t)B)x, x \rangle + \langle f((1-t)A + tB)x, x \rangle] dt \\
& \leq \frac{1}{4} \int_0^1 [\langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle \\
& \quad + \langle f((1-t)A + tB)x, x \rangle \langle g((1-t)A + tB)x, x \rangle] dt \\
& + \frac{1}{4} M(A, B)(x) \int_0^1 [2t(1-t)] dt + \frac{1}{4} N(A, B)(x) \int_0^1 [t^2 + (1-t)^2] dt \\
& + \left\langle f \left(\frac{A+B}{2} \right) x, x \right\rangle \left\langle g \left(\frac{A+B}{2} \right) x, x \right\rangle \int_0^1 dt.
\end{aligned}$$

Then we have the required inequality (2.9) □

3. APPLICATION FOR SYNCHRONOUS (ASYNCHRONOUS) FUNCTIONS

We say that the functions $f, g : [a, b] \rightarrow \mathbb{R}$ are synchronous (asynchronous) on the interval $[a, b]$ if they satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq) 0 \text{ for each } t, s \in [a, b]$$

It is obvious that, if f, g are monotonic and have the same monotonicity on the interval $[a, b]$, then they are synchronous on $[a, b]$ while if they have opposite monotonicity, they are asynchronous.

The following result provides an inequality of Čebyšev type for functions of selfadjoint operators.

Theorem 6. (Dragomir,[1]) *Let A be a selfadjoint operator with $Sp(A) \subset [m, M]$ for some real numbers $m < M$, If $f, g : [m, M] \rightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then*

$$(3.1) \quad \langle f(A)g(A)x, x \rangle \geq (\leq) \langle f(A)x, x \rangle \langle g(A)x, x \rangle,$$

for any $x \in H$ with $\|x\| = 1$.

As a simple consequence of the above Theorem we imply that if f, g are synchronous, then

$$(3.2) \quad N(A, B)(x) \leq M(A, B)(x) \leq P(A, B)(x),$$

for any $x \in H$ with $\|x\| = 1$. If f, g are asynchronous, then reverse inequalities holds in (3.2) as follow,

$$(3.3) \quad N(A, B)(x) \geq M(A, B)(x) \geq P(A, B)(x).$$

Remark 1. Let $f, g : [m, M] \rightarrow \mathbb{R}$ be operator convex and A, B be selfadjoint operator with $Sp(A) \cup sp(B) \subset [m, M]$

- (i) If f, g are synchronous and $f, g \geq 0$ then the inequality (2.2) becomes

$$(3.4) \quad \int_0^1 \langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle dt \\ \leq \frac{1}{3}M(A, B)(x) + \frac{1}{6}N(A, B)(x) \leq \frac{1}{2}P(A, B)(x).$$

If f, g are synchronous then the inequalities (2.7) and (2.9) become

$$(3.5) \quad \left\langle f\left(\frac{A+B}{2}\right)x, x \right\rangle \left\langle g\left(\frac{A+B}{2}\right)x, x \right\rangle \\ \leq \frac{1}{2} \int_0^1 \langle f(tA + (1-t)B)g(tA + (1-t)B)x, x \rangle dt \\ + \frac{1}{4}P(A, B)(x),$$

and

$$\begin{aligned}
 (3.6) \quad & \left\langle f \left(\frac{A+B}{2} \right) x, x \right\rangle \int_0^1 \langle g(tA + (1-t)B)x, x \rangle dt \\
 & + \left\langle g \left(\frac{A+B}{2} \right) x, x \right\rangle \int_0^1 \langle f(tA + (1-t)B)x, x \rangle dt \\
 & \leq \frac{1}{2} \int_0^1 \langle f(tA + (1-t)B)g(tA + (1-t)B)x, x \rangle dt \\
 & \quad + \frac{1}{4}P(A, B)(x) + \left\langle f \left(\frac{A+B}{2} \right) g \left(\frac{A+B}{2} \right) x, x \right\rangle.
 \end{aligned}$$

(ii) If f, g are asynchronous and $f, g \geq 0$ then the inequality (2.2) becomes

$$\begin{aligned}
 (3.7) \quad & \int_0^1 \langle f(tA + (1-t)B)g(tA + (1-t)B)x, x \rangle dt \\
 & \leq \frac{1}{3}M(A, B)(x) + \frac{1}{6}N(A, B)(x) \leq \frac{1}{2}N(A, B)(x).
 \end{aligned}$$

If f, g are synchronous then the inequalities (2.7) and (2.9) become

$$\begin{aligned}
 (3.8) \quad & \left\langle f \left(\frac{A+B}{2} \right) g \left(\frac{A+B}{2} \right) x, x \right\rangle \\
 & \leq \frac{1}{2} \int_0^1 \langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle dt \\
 & \quad + \frac{1}{4}N(A, B)(x)
 \end{aligned}$$

and

$$\begin{aligned}
 (3.9) \quad & \left\langle \left[f \left(\frac{A+B}{2} \right) \int_0^1 g(tA + (1-t)B)dt \right. \right. \\
 & \quad \left. \left. + g \left(\frac{A+B}{2} \right) \int_0^1 f(tA + (1-t)B)dt \right] x, x \right\rangle \\
 & \leq \frac{1}{2} \int_0^1 \langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle dt \\
 & \quad + \frac{1}{4}N(A, B)(x) + \left\langle f \left(\frac{A+B}{2} \right) x, x \right\rangle \left\langle g \left(\frac{A+B}{2} \right) x, x \right\rangle.
 \end{aligned}$$

Example 1. Real functions $f(x) = x$ and $g(x) = x^2$ are synchronous on $[0, 1]$. From inequality (3.4) we get the inequality

$$\int_0^1 \langle (tA + (1-t)B)x, x \rangle \langle (tA + (1-t)B)^2x, x \rangle dt \leq \frac{1}{2}P(A, B)(x),$$

which holds for any selfadjoint operators A and B on a Hilbert space H with spectra in $[0, 1]$.

Real functions f, g are asynchronous on $[-1, 0]$, Now from inequality (3.8) we get the inequality

$$\begin{aligned} & \left\langle \left(\frac{A+B}{2} \right)^3 x, x \right\rangle \\ & \leq \frac{1}{2} \int_0^1 \langle (tA + (1-t)B)x, x \rangle \langle (tA + (1-t)B)^2x, x \rangle dt \\ & \quad + \frac{1}{4}N(A, B)(x) \end{aligned}$$

which holds for any selfadjoint operators A and B on a Hilbert space H with spectra in $[-1, 0]$, where

$$\begin{aligned} M(A, B)(x) &= \langle Ax, x \rangle \langle A^2x, x \rangle + \langle Bx, x \rangle \langle B^2x, x \rangle, \\ N(A, B)(x) &= \langle Ax, x \rangle \langle B^2x, x \rangle + \langle Bx, x \rangle \langle A^2x, x \rangle, \\ P(A, B)(x) &= \langle (A^3 + B^3)x, x \rangle. \end{aligned}$$

We may obtain other inequalities which follow from (3.5), (3.6), (3.7) and (3.9), the details are omitted.

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